Solving linear ordinary differential equations over $C(x, e^{\int f(x)dx})$

Manuel Bronstein
INRIA – Projet CAFÉ
2004 Route des Lucioles, BP 93
F-06902 Sophia Antipolis Cedex
Manuel.Bronstein@sophia.inria.fr

Anne Fredet
Laboratoire GAGE
Ecole Polytechnique
F-91128 Palaiseau Cedex
fredet@ariana.polytechnique.fr

Abstract

We describe a new algorithm for computing the solutions in $F = C(x, e^{\int f(x)dx})$ of linear ordinary differential equations with coefficients in F. Compared to the general algorithm of [9], our algorithm avoids the computation of exponential solutions of equations with coefficients in $C(x)$, as well as the solving of linear differential systems over $C(x)$. Our method is effective and has been implemented.

1 Introduction

We recall that a differential field is a field K together with a map $' : K \to K$ satisfying $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for any $a, b \in K$. In that case the constant field of K is defined as

$$\text{Const}(K) = \{a \in K \text{ s.t. } a' = 0\}.$$

All fields in this paper are of characteristic 0.

Definition 1 *Let K be a differential field and $L = \sum_{i=0}^{n} a_i \partial^i$ be a linear ordinary differential operator with coefficients in K. A* rational solution *of L is any element $y \in K$ such that $Ly = 0$, and an* exponential solution *of L is any element $u \in K$ such that $Le^{\int u} = 0$.*

Although our main goal is to compute the rational solutions of such equations, it turns out to be more useful to solve the following, slightly more general problem, instances of which are generated by our algorithm even when starting with a homogeneous equation.

Definition 2 *We say that we can* effectively solve parameterized linear differential equations over K *if given any linear ordinary differential operator $L = \sum_{i=0}^{n} a_i \partial^i$ with coefficients in K and $g_1, \ldots, g_m$ in K, we can effectively find $h_1, \ldots, h_r$ in K and a matrix M with $r + m$ columns and coefficients in Const(K) such that $Ly = \sum_{j=1}^{m} c_j g_j$ for $y \in K$ and $c_1, \ldots, c_m \in$ Const(K) if and only if $y = \sum_{k=1}^{r} y_k h_k$ where $y_1, \ldots, y_r \in$ Const(K) and $M(y_1, \ldots, y_r, c_1, \ldots, c_m)^T = 0$.*

In [9], Singer gave an algorithm that reduced the problem of solving parameterized linear differential equations over a Liouvillian extension $E = K(t)$ of K to solving such equations over K *and,* in the case where t is exponential over K, to compute all the exponential solutions of linear ordinary differential equations over K. His method, although algorithmic, suffers from several difficulties, mainly the requirement to compute exponential solutions in Liouvillian towers, and the reduction to linear differential systems rather than scalar equations once the order and degree of (Laurent)–polynomial solutions are bounded. To our knowledge, that method has not been implemented, even for the case where $K = \mathbb{Q}(x)$. In this paper, we describe an alternative algorithm that does not require computing exponential solutions over $C(x)$ when E is of the form $C(x, e^{\int f(x)dx})$, $' = d/dx$, and E is not an algebraic function field. Furthermore, for transcendental exponential extensions of an arbitrary differential field K, our algorithm directly produces scalar equations over K (without increasing the order) rather than differential systems.

Let now C be an arbitrary field[1], x be transcendental over C and $E = C(x,t)$ be a differential extension of $(C(x), d/dx)$ where $dt/dx = ft$ for some $f \in C(x)$, *i.e.* $t = e^{\int f(x)dx}$. We write $'$ for d/dx throughout this paper. There are three separate steps in solving the parameterized equation $Ly = \sum_{j=1}^{m} c_j g_j$ for $y \in E$ and $c_1, \ldots, c_m \in C$:

1. compute the "denominator" of y, *i.e.* compute $q \in C(x)[t]$ such that $qy \in C(x)[t, t^{-1}]$ for any solution y. Replacing y by Y/q in the parameterized equation, this reduces the problem to solving a similar equation for $Y \in C(x)[t, t^{-1}]$;
2. bound the degree and order of Y, *i.e.* compute integers $\delta \le \gamma$ such that any solution $Y \in C(x)[t, t^{-1}]$ must be of the form
$$Y = y_\delta t^\delta + \ldots + y_\gamma t^\gamma \quad \text{where } y_\delta, \ldots, y_\gamma \in C(x),$$
3. compute the solution Y knowing δ and γ.

In the case when t is transcendental over $C(x)$ and the constant subfield of E is C (we will eventually ensure those conditions), then the first step is done using either Lemma 3.2 of [9] or Theorem 5.5 of [3], so we address only the last

[1] For the algorithm to be complete, we eventually require that a vector space basis for C over $\mathbb{Q}$ be known.

two steps in this paper. The third step is done in [9] by replacing Y by $\sum_{i=\delta}^{\gamma} y_i t^i$ in the equation and computing the rational solutions of the resulting linear differential system for the y_i's.

The rest of this paper is organized as follows: some preliminary results about exponential extensions are given in section 2, then a specialized bounding algorithm for exponential extensions of $C(x)$ is described in section 3, and a new algorithm for the third step in arbitrary exponential extensions is described in section 4.

Notation: for any field k, Z transcendental over k, and any $f \in k(Z)^*$, let $\sum_{i\geq 0} a_i Z^{d+i}$ and $\sum_{j\geq 0} b_j Z^{e-j}$ be the Laurent expansions of f at $Z=0$ and $Z=\infty$ respectively where $a_0 \neq 0 \neq b_0$. We write $d = \nu_Z(f)$ (the order of f), $e = \deg_Z(f)$ (the degree of f) and $b_0 = \mathrm{lc}_Z(f)$ (the leading coefficient of f). For any nonzero linear ordinary differential operator with coefficients in $k(Z)$, $L = f_n\partial^n + f_{n-1}\partial^{n-1} + \ldots + f_0$, we write $\deg_Z(L) = \max_{0\leq i\leq n}(\deg_Z(f_i))$ and $\nu_Z(L) = \min_{0\leq i\leq n}(\nu_Z(f_i))$. We omit the subscripts when the context is clear (to prevent confusion, we never use $\deg(L)$ for the order of L).

2 Preliminaries

We group in this section some known results about exponential extensions that will be used by our algorithm. Let $(K,')$ be an arbitrary differential field.

Lemma 1 *For any $a_0,\ldots,a_n,a,b \in K$ with $b\neq 0$,*

$$\left(\sum_{i=0}^{n} a_i\partial^i\right)(ab) = b\left(\sum_{i=0}^{n} a_i\left(\partial + \frac{b'}{b}\right)^i\right)a\,.$$

Proof. The formula is trivial for $n=0$, so assume it holds for operators of order up to $n-1$. We have

$$\begin{aligned} b\left(\sum_{i=0}^{n} a_i\left(\partial+\frac{b'}{b}\right)^i\right)a &= ba_0a + b\left(\sum_{i=0}^{n-1} a_{i+1}\left(\partial+\frac{b'}{b}\right)^i\right)\left(a'+\frac{b'}{b}a\right)\\ &= a_0ab + \left(\sum_{i=0}^{n-1} a_{i+1}\partial^i\right)(a'b) + \left(\sum_{i=0}^{n-1} a_{i+1}\partial^i\right)(ab')\\ &= \left(\sum_{i=0}^{n} a_i\partial^i\right)(ab)\,. \end{aligned}$$

□

Lemma 2 *Let E be a differential extension of K and $t\in E^*$ be such that $t'/t \in K$. Then, for any linear ordinary differential operator L with coefficients in K,*

$$\frac{L(at^m)}{t^m} \in K \quad \text{for any integer } m \text{ and any } a\in K\,.$$

Proof. Write $L = \sum_{i=0}^{n} a_i\partial^i$ where the a_i's are in K. By Lemma 1,

$$\begin{aligned} L(at^m) &= \left(\sum_{i=0}^{n} a_i\partial^i\right)(at^m)\\ &= t^m\left[\left(\sum_{i=0}^{n} a_i\left(\partial + m\frac{t'}{t}\right)^i\right)a\right] = t^m b \end{aligned}$$

where $b\in K$ since $t'/t\in K$. □

We now introduce the construction of "projected" operators: let E be a differential extension of K and $t\in E$ be transcendental over K. We can uniquely decompose any linear ordinary differential operator L with coefficients in $K[t]$ as

$$L = \sum_{k=\nu(L)}^{\deg(L)} t^k L_k \qquad (1)$$

where the L_k's are linear ordinary differential operators with coefficients in K (the "projections" of L with respect to t). Note that $L_{\nu(L)} \neq 0 \neq L_{\deg(L)}$.

The following, whose proof is present in [9], is the basis of all the algorithms for bounding the order and degree of (Laurent)–polynomial solutions.

Proposition 1 *Let E be a differential extension of K, $t\in E$ be transcendental and exponential over K, L be a linear ordinary differential operator with coefficients in $K[t]$ and $b_1,\ldots,b_m \in K(t)$. If there are integers $\delta\leq\gamma$, $y_\delta,\ldots,y_\gamma \in K$ and $c_1,\ldots,c_m\in Const(K)$ such that $y_\delta\neq 0\neq y_\gamma$ and*

$$L\left(y_\delta t^\delta + \ldots + y_\gamma t^\gamma\right) = c_1b_1 + \ldots + c_mb_m\,, \qquad (2)$$

then,

(i) either $\delta \geq \min_{1\leq j\leq m}(\nu(b_j)) - \nu(L)$ or $L_{\nu(L)}(y_\delta t^\delta) = 0$;

(ii) either $\gamma \leq \max_{1\leq j\leq m}(\deg(b_j)) - \deg(L)$ or $L_{\deg(L)}(y_\gamma t^\gamma) = 0$;

where the minimum and maximum are taken over the nonzero b_j's.

Proof. Using the decomposition (1) we have

$$\begin{aligned} L\left(y_\delta t^\delta + \ldots + y_\gamma t^\gamma\right) &= \sum_{k=\nu(L)}^{\deg(L)} t^k L_k\left(\sum_{i=\delta}^{\gamma} y_i t^i\right)\\ &= \sum_{k=\nu(L)}^{\deg(L)}\sum_{i=\delta}^{\gamma} z_{ki}\,t^{k+i} \qquad (3) \end{aligned}$$

where $z_{ki} = L_k(y_it^i)/t^i$ is in K by Lemma 2.
(i) Suppose that $\delta < \min_{1\leq j\leq m}(\nu(b_j)) - \nu(L)$. It then follows from (2) and (3) that $z_{\nu(L)\delta} = 0$, hence that $L_{\nu(L)}(y_\delta t^\delta) = 0$.
(ii) Suppose that $\gamma > \max_{1\leq j\leq m}(\deg(b_j)) - \deg(L)$. It then follows from (2) and (3) that $z_{\deg(L)\delta} = 0$, hence that $L_{\deg(L)}(y_\gamma t^\gamma) = 0$. □

This reduces the problem to bounding the integers m such that $L(at^m) = 0$ for $a\in K^*$ and L an operator with coefficients in K. This problem is solved for the case $K = C(x)$ and $' = d/dx$ in the next section.

3 Bounding the order and degree

We now consider the case $E = C(x,t)$ where $' = d/dx$ and $t'/t \in C(x)$, *i.e.* $t = e^{\int f(x)dx}$. Since we do not consider algebraic extensions in this paper, we make the additional assumption that t'/t is not the logarithmic derivative of a $C(x)$-radical, *i.e.* t'/t cannot be written as $n^{-1}f'/f$ for $f\in C(x)$ and n a nonzero integer. By [4, Theorem 5.1.2], this implies that t is transcendental over $C(x)$ and that the constant field of E is C. Let $u\in C(x)$ and $y = e^{\int u dx}$. Then,

$y^{(i)} = P_i(u, u', \ldots, u^{(i-1)})y$ where the P_i's are polynomials with integer coefficients given by

$$P_0 = 1 \quad \text{and} \quad P_i = P_{i-1}' + uP_{i-1} \quad \text{for } i > 0\,. \tag{4}$$

Let $L = \sum_{i=0}^{n} a_i \partial^i$ be a linear ordinary differential operator with coefficients in $C(x)$. If $Le^{\int u\,dx} = 0$, then

$$a_n P_n(u, \ldots, u^{(n-1)}) + a_{n-1} P_{n-1}(u, \ldots, u^{(n-2)}) + \ldots + a_0 = 0\,,$$

and the above equation is called the *Riccati equation associated with* L. The following two Lemmas are special cases of Lemma 2.2 of [9] and their proofs can be found there. For the second one, we use that $P_i(c, \ldots, c^{(i-1)}) = c^i$ for $c \in C$.

Lemma 3 ([9]) *Let $p \in C[x]$ be irreducible, and $u \in C(x)$ have p-adic expansion of the form $u = u_\gamma/p^\gamma + \ldots$ where $\gamma > 0$, $u_\gamma \neq 0$ and $\deg u_\gamma < \deg p$. Then, for $i \geq 1$ the p-adic expansion of $P_i(u, \ldots, u^{(i-1)})$ has the form $v_{i\gamma}/p^{i\gamma} + \ldots$ where*

$$\begin{cases} v_{i\gamma} \equiv u_\gamma^i & (\mathrm{mod}\ p) \quad \text{if } \gamma > 1\,, \\ v_{i\gamma} \equiv \prod_{j=0}^{i-1}(u_1 - jp') & (\mathrm{mod}\ p) \quad \text{if } \gamma = 1\,. \end{cases}$$

Lemma 4 ([9]) *Let $u \in C(x)$ have a Laurent expansion at $x = \infty$ of the form $u = \sum_{i \geq 0} u_i x^{\gamma - i} + \ldots$ where $\gamma \geq 0$ and $u_0 \neq 0$. Then, for $i \geq 1$ the Laurent expansion at $x = \infty$ of $P_i(u, \ldots, u^{(i-1)})$ has the form $u_0^i x^{i\gamma} + \ldots$.*

During the bounding process, we may have to compute the integer solutions of some algebraic equations. The following Lemma shows that the systems that arise have finitely many such solutions.

Lemma 5 *Let $K \subset F$ be fields and $p \in F[Z]$ be irreducible with $\deg(p) = m$. Let $a_0, \ldots, a_n \in F[Z]$ and $q, u \in F[Z]$ be linearly independent over K and such that $\deg(q) < m$ and $\deg(u) < m$. If $p \not| \gcd(a_0, \ldots, a_n)$, then the set*

$$S = \{(\gamma, \alpha) \in K^2 \text{ such that } \sum_{i=0}^{n} a_i(\gamma u + \alpha q)^i \equiv 0 \pmod{p}\}$$

is finite with at most n elements.

Proof. Let $\pi : F[Z] \to F[Z]/(p)$ be the canonical projection and $Q = \sum_{i=0}^{n} \pi(a_i) T^i \in (F[Z]/(p))[T]$. Since $p \not| \gcd(a_0, \ldots, a_n)$, Q is not identically 0 and has degree at most n, so let $\beta_1, \ldots, \beta_k$ with $k \leq n$ be its distinct roots in $F[Z]/(p)$, and let $p_1, \ldots, p_k \in F[Z]$ be such that $\deg(p_j) < m$ and $\pi(p_j) = \beta_j$ for each j. For any $(\gamma, \alpha) \in S$, we have $Q(\gamma\pi(u) + \alpha\pi(q)) = 0$, which implies that $\pi(\gamma u + \alpha q) = \pi(p_j)$ for some j, hence that $\gamma u + \alpha q = p_j$ since the degrees of u, q and p_j are strictly less than m. But for each j, there is at most one pair $(a, b) \in K^2$ such that $au + bq = p_j$, since the difference of two such relations with distinct pairs would yield a linear dependence over K for u and q. Therefore, S has at most n elements. □

We now return to bounding the integers h such that $L(at^h) = 0$ for $a \in C(x)^*$ and any operator L with coefficients in $C(x)$. For exponential extensions of $C(x)$, we replace the search for exponential solutions by the following adaptation of the algorithm of Proposition 2.3 of [9]. Our modification computes only the potential degrees of "monomial" solutions rather than all the exponential solutions.

Proposition 2 *Suppose that $E = C(x, t)$ where $' = d/dx$, $t'/t \in C(x)$ is not the logarithmic derivative of a $C(x)$-radical, that we know a vector space basis for C over $\mathbb{Q}$, that we can factor polynomials in $C[Z]$ and compute their integer roots. Then, for any nonzero linear ordinary differential operator L with coefficients in $C(x)$, we can compute a finite set $\beta(L) \subset \mathbb{Z}$ such that if $L(at^\gamma) = 0$ for an integer γ and $a \in C(x)^*$, then $\gamma \in \beta(L)$.*

Proof. Write

$$L = A_n \partial^n + \ldots + A_0 \quad \text{where } A_i \in C[x], A_n \neq 0\,,$$

and

$$\frac{t'}{t} = G + \frac{A}{D}$$

where $G, A, D \in C[x]$, $D \neq 0$ and $\deg(A) < \deg(D)$. Suppose that $L(at^\gamma) = 0$ for some $a \in C(x)^*$ and an integer γ. By adding 0 to $\beta(L)$, we can assume without loss of generality that $\gamma \neq 0$. We have

$$\sum_{i=0}^{n} A_i P_i(u, u', \ldots, u^{(i-1)}) = 0 \tag{5}$$

where $u = a'/a + \gamma t'/t$ and the P_i's are given by (4). Note that for any irreducible $p \in C[x]$, the p-adic expansion of a'/a is $a'/a = \nu p' p^{-1} + \ldots$ where $\nu \in \mathbb{Z}$ is the order of a at p. We distinguish several cases.

1. If $G \neq 0$, then the Laurent expansion of t'/t at $x = \infty$ has the form $\mathrm{lc}(G)x^M + \ldots$ where $M = \deg(G)$, so the expansion of u is $\gamma \mathrm{lc}(G)x^M + \ldots$, and by Lemma 4, the expansion of $P_i(u, u', \ldots, u^{(i-1)})$ is $\gamma^i \mathrm{lc}(G)^i x^{iM} + \ldots$. Therefore, the expansion of the left-hand side of (5) is

$$\left(\sum_{i \text{ s.t. } \deg(A_i) + iM = \mu} \mathrm{lc}(A_i)\gamma^i \mathrm{lc}(G)^i \right) x^\mu + \ldots$$

where $\mu = \max_i(\deg(A_i) + iM)$. It follows that $Q(\gamma) = 0$ where

$$Q = \sum_{i \text{ s.t. } \deg(A_i) + iM = \mu} \mathrm{lc}(A_i)\mathrm{lc}(G)^i Z^i \quad \in C[Z]$$

and $\beta(L)$ is the set of all the integer roots of Q.

2. If there is an irreducible $p \in C[x]$ such that $p^2 \mid D$, then let $t'/t = u_\beta/p^\beta + \ldots$ be the p-adic expansion of t'/t where $\beta \geq 2$ and $u_\beta \in C[x]$ is nonzero and has degree strictly less than $\deg(p)$. The p-adic expansion of u is then $\gamma u_\beta/p^\beta + \ldots$, so by Lemma 3, the p-adic expansion of $P_i(u, u', \ldots, u^{(i-1)})$ is $v_{i\beta}/p^{i\beta} + \ldots$ where $v_{i\beta} \equiv \gamma^i u_\beta^i \pmod{p}$. Let

$$A_i = a_{\alpha_i} p^{\alpha_i} + \ldots \tag{6}$$

with $\alpha_i \geq 0$ be the p-adic expansion of each nonzero A_i and $\mu = \min_i(\alpha_i - i\beta)$. Then, the p-adic expansion of the left-hand side of (5) is

$$\left(\sum_{i \text{ s.t. } \alpha_i - i\beta = \mu} w_i \right) p^\mu + \ldots$$

where $w_i \equiv a_{\alpha_i}\gamma^i u_\beta^i \pmod p$. It follows that $Q(\gamma) \equiv 0 \pmod p$ where

$$Q = \sum_{i \ s.t. \ \alpha_i - i\beta = \mu} a_{\alpha_i} u_\beta^i Z^i \quad \in C[x][Z] \,.$$

Computing the remainder of Q by p in $C[Z][x]$ (the leading coefficient of p being a unit in C[Z]) and equating each coefficient of x^j to 0, we get a system of polynomial equations $Q_0(\gamma) = \ldots = Q_{\deg(p)-1}(\gamma) = 0$, hence, $H(\gamma) = 0$ where $H = \gcd(Q_0, \ldots, Q_{\deg(p)-1}) \in C[Z]$. Note that H is not identically 0, since the leading coefficient of Q is $a_{\alpha_I} u_\beta^I \in (C[x]/(p))^*$ where $I = \max\{i \text{ s.t. } \alpha_i - i\beta = \mu\}$. As earlier, $\beta(L)$ is the set of all the integer roots of Q.

3. If there is an irreducible $p \in C[x]$ such that $p \mid D$, $p^2 \not| D$ and u_1 and p' are linearly independent over $\mathbb{Q}$, where $u_1 \in C[x] \setminus \{0\}$ is given by $\deg(u_1) < \deg(p)$ and $t'/t = u_1 p^{-1} + \ldots$ is the p-adic expansion of t'/t. Then, the p-adic expansion of u is $(\gamma u_1 + \nu p')p^{-1} + \ldots$, where $\nu \in \mathbb{Z}$ (note that ν can be 0 if p does not divide the numerator or denominator of a). By Lemma 3, the p-adic expansion of $P_i(u, u', \ldots, u^{(i-1)})$ is $v_i/p^i + \ldots$ where $v_i \equiv \prod_{j=0}^{i-1}(\gamma u_1 + \nu p' - jp') \pmod p$. Using (6), the p-adic expansion of the left-hand side of (5) is

$$\left(\sum_{i \ s.t. \ \alpha_i - i = \mu} w_i \right) p^\mu + \ldots$$

where $\mu = \min_i(\alpha_i - i)$ and $w_i \equiv a_{\alpha_i} \prod_{j=0}^{i-1}(\gamma u_1 + \nu p' - jp') \pmod p$. It follows that $Q(\gamma u_1 + \nu p') \equiv 0 \pmod p$ where

$$Q = \sum_{i \ s.t. \ \alpha_i - i = \mu} a_{\alpha_i} \prod_{j=0}^{i-1} (Z - jp') \quad \in C[x][Z] \,.$$

As in the previous case, computing the remainder of Q by p in $C[Z][x]$ and equating it to 0, we get a system of polynomial equations $Q_0(\gamma, \nu) = \ldots = Q_{\deg(p)-1}(\gamma, \nu) = 0$ with coefficients in C. If that system has dimension 0, then we can compute (via resultants or Gröbner bases) a nontrivial univariate equation $H(\gamma) = 0$ where $H \in C[Z]$ is in the ideal generated by $Q_0, \ldots, Q_{\deg(p)-1}$. In that case, $\beta(L)$ is as earlier the set of all the integer roots of H. If the system has a strictly positive dimension[2], we then write $Q = \sum_{i=0}^{n} Q_i Z^i$ where $Q_0, \ldots, Q_n \in C[x]$. Since the leading coefficient of Q is a_{α_I} where $I = \max\{i \text{ s.t. } \alpha_i - i = \mu\}$ and $\deg(a_{\alpha_I}) < \deg(p)$, it follows that $p \not| \gcd(Q_0, \ldots, Q_n)$, so by Lemma 5, the equation $Q(\gamma u_1 + \nu p') \equiv 0 \pmod p$ has a finite set of rational solutions, which we can compute in the following way: since we can factor elements of $C[Z]$, we can also factor elements of $(C[x]/(p))[Z]$ [7, §8.8], so the roots $\beta_1, \ldots, \beta_k$ of Q in $C[x]/(p)$ can be computed from its linear factors over $C[x]/(p)$. For each β_j, equating the powers of x on both sides of the equation $\gamma u_1 + \alpha p' = \beta_j$ yields a linear system $A_j(\gamma, \alpha)^T = \mathbf{v_j}$ with coefficients in C for γ and α. Projecting that system on our basis for C over $\mathbb{Q}$, we get a matrix B_j and a vector $\mathbf{w_j}$, both with coefficients in $\mathbb{Q}$ such that the rational solutions of $A_j(\gamma, \alpha)^T = \mathbf{v_j}$ and $B_j(\gamma, \alpha)^T = \mathbf{w_j}$ are the same. If B_j is nonsingular, then that system has a unique solution, which we compute. Otherwise, that system has no solution since its set of rational solutions is finite. Selecting the integer solutions from the rational ones yields the set $\beta(L)$.

[2] According to Lemma 5 this can happen only when u_1 and p' are linearly dependent over C.

Suppose that none of the above cases is applicable, then $t'/t = A/D$ where $\deg(A) < \deg(D)$ and every irreducible factor p of D must satisfy the following conditions: $p^2 \not| D$, and $\{u_1, p'\}$ are linearly dependent over $\mathbb{Q}$, where u_1 is defined as in the third case. This linear dependence implies that for each such p, there is a rational number $r_p \in \mathbb{Q}$ such that p does not divide the denominator of $t'/t - r_p p'/p$. Therefore, $\eta = t'/t - \sum_{p|D} r_p p'/p \in C[x]$, where the sum is taken over all the irreducible factors of D. But the polynomial part of η is 0, so $\eta = 0$. Writing $r_p = n_p/m$ for a common denominator m, we get $t'/t = m^{-1} f'/f$ where $f = \prod_{p|D} p^{n_p} \in C(x)^*$, in contradiction with t'/t not being the logarithmic derivative of a $C(x)$-radical. □

Theorem 1 *Suppose that $E = C(x,t)$ where $' = d/dx$, $t'/t \in C(x)$ is not the logarithmic derivative of a $C(x)$-radical, that we know a vector space basis for C over $\mathbb{Q}$, that we can factor polynomials in $C[Z]$ and compute their integer roots. Then, for any nonzero linear ordinary differential operator L with coefficients in E and $b_1, \ldots, b_m \in E$, we can compute finite sets $\Delta \subset \mathbb{Z}$ and $\Gamma \subset \mathbb{Z}$ such that if there are integers $\delta \le \gamma$, $y_\delta, \ldots, y_\gamma \in C(x)$ and $c_1, \ldots, c_m \in C$ such that $y_\delta \neq 0 \neq y_\gamma$ and*

$$L\left(y_\delta t^\delta + \ldots + y_\gamma t^\gamma\right) = c_1 b_1 + \ldots + c_m b_m \,,$$

then $\delta \in \Delta$ and $\gamma \in \Gamma$.

Proof. This follows immediately from Propositions 1 and 2. □

4 Computing the solutions

We now turn back to the third step, namely computing the solutions once bounds on their degree and order are known. Throughout this section, let K be an arbitrary differential field and t be transcendental and exponential over K. Suppose that $\mathrm{Const}(K(t)) = \mathrm{Const}(K)$ and that we can effectively solve parameterized linear differential equations over K. Let L be a linear ordinary differential operator with coefficients in $K[t]$, $b_1, \ldots, b_m \in K(t)$ and integers $\delta \le \gamma$ be given. We look for $y_\delta, \ldots, y_\gamma \in K$ and $c_1, \ldots, c_m \in \mathrm{Const}(K)$ such that $L\left(\sum_{i=\delta}^{\gamma} y_i t^i\right) = \sum_{j=1}^{m} c_j b_j$. Let $z = t^{-\delta} y$. We have $z \in K[t]$, $\deg(z) \le N = \gamma - \delta$ and, by Lemma 1, $Ly = L(t^\delta z) = t^\delta \tilde{L} z$ where $\tilde{L}$ is also an operator with coefficients in $K[t]$. It is thus sufficient to look for solutions $p \in K[t]$ of degree at most $N \ge 0$ of $Lp = \sum_{j=1}^{m} c_j b_j$. Dividing both sides by a power of t if necessary, we can assume that $\nu(L) = 0$. Let $d \in K[t]$ be a common denominator for the b_i's and write $b_i = f_i + g_i/d$ where $f_i, g_i \in K[t]$ and $\deg(g_i) < \deg(d)$. If $Lp = \sum_{j=1}^{m} c_j b_j$ for $p \in K[t]$, then

$$\frac{\sum_{j=1}^{m} c_j g_j}{d} = Lp - \sum_{j=1}^{m} c_j f_j \quad \in K[t]$$

which implies that $Lp = \sum_{j=1}^m c_j f_j$ and that $\sum_{j=1}^m c_j g_j = 0$. Equating the coefficients of each t^i to 0 in that last condition yields a matrix A with coefficients in K such that $A(c_1, \ldots, c_m)^T = 0$. Using Lemma 7.1.2 of [4] and the associated algorithm, we construct a matrix B with coefficients in $\mathrm{Const}(K)$ such that the constant solutions of $A\mathbf{x} = 0$ and $B\mathbf{x} = 0$ are the same.

We are reduced to solving $Lp = \sum_{j=1}^m c_j f_j$ where $p, f_1, \ldots, f_m \in K[t]$ and $\deg(p) \leq N$ for a given N.

If $N = 0$, then $p \in K$, so using the decomposition (1) our equation becomes

$$\sum_{k=0}^{\deg(L)} t^k L_k p = \sum_{j=1}^m c_j f_j$$

and equating the coefficients of each t^k on both sides we get either new linear constraints for the c_j's (if $k > \deg(L)$ or $L_k = 0$), or parameterized linear differential equations over K for p (if $L_k \neq 0$), which we can solve using the induction hypothesis.

If $N > 0$, write $p = p_0 + tq$ where $p_0 = p(0) \in K$ and $q \in K[t]$ with $\deg(q) < N$. Since $t'/t \in K$, the ideal (t) of $K[t]$ is closed under $'$, so the canonical projection $\pi : K[t] \to K = K[t]/(t)$, which sends $h \in K[t]$ to $h(0)$ is a differential homomorphism [4, Theorem 4.2.1]. Applying it to our equation we get $L_0 p_0 = \sum_{j=1}^m c_j f_j(0)$ where $L_0 = L_{\nu(L)}$ is given by the decomposition (1). It is a nonzero linear ordinary differential operator with coefficients in K, so using the induction hypothesis we compute $h_1, \ldots, h_r \in K$ and a matrix M with coefficients in $\mathrm{Const}(K)$ such that for any solution $p_0 \in K$ we must have $p_0 = \sum_{i=1}^r e_i h_i$ and $M(e_1, \ldots, e_r, c_1, \ldots, c_m)^T = 0$. Replacing p by $\sum_{i=1}^r e_i h_i + tq$ in the equation, we get

$$\begin{aligned}\sum_{j=1}^m c_j f_j &= Lp = Lp_0 + L(tq) \\ &= \sum_{j=1}^m c_j f_j(0) + (L - L_0)\left(\sum_{i=1}^r e_i h_i\right) + t\tilde{L}q\end{aligned}$$

where $\tilde{L}$ is given by Lemma 1. Therefore,

$$\tilde{L}q = \sum_{j=1}^m c_j \frac{f_j - f_j(0)}{t} - \sum_{i=1}^r e_i \frac{1}{t}(L - L_0)h_i\,.$$

The above is a parameterized equation with right–hand side in $K[t]$ and we solve it with the bound $N-1$ for $\deg(q)$ using the same method. This process eventually terminates since the degree bound is reduced at each step.

We remark that our algorithm becomes simpler when $L = \sum_k a_k \partial^k$ has its coefficients $a_k \in K$. In that case any solution $y = \sum_{i=\delta}^\gamma y_i t^i$ of $Ly = \sum_{j=1}^m c_j f_j$ where $f_j \in K[t, t^{-1}]$ satisfies

$$L\left(\sum_{i=\delta}^\gamma y_i t^i\right) = \sum_{i=\delta}^\gamma z_i t^i = \sum_{j=1}^m c_j f_j$$

where $z_i = L(y_i t^i)/t^i \in K$ by Lemma 2. Writing $f_j = \sum_i f_{ji} t^i$ for $f_{ji} \in K$, we get the equations

$$\left(\sum_k a_k \left(\partial + i\frac{t'}{t}\right)^k\right) y_i = \sum_{j=1}^m c_j f_{ji} \quad \text{for } \delta \leq i \leq \gamma \qquad (7)$$

as well as the linear constraints

$$\sum_{j=1}^m c_j f_{ji} = 0 \quad \text{for } i < \delta \text{ or } i > \gamma\,.$$

Those constraints can be reduced to linear constraints with constant coefficients as earlier, and the equations (7) can be solved independently of each other. In the homogeneous case and when the coefficients of L are in K, then any solution $y \in K(t)$ of $Ly = 0$ must be in $K[t, t^{-1}]$ [3, Theorem 5.5], and the equations (7) become

$$\left(\sum_k a_k \left(\partial + i\frac{t'}{t}\right)^k\right) y_i = 0$$

so parametric equations are not introduced, unlike when t appears in the coefficients of L. Furthermore, if $K = C(x)$ and $' = d/dx$, then the set $\beta(L)$ of Proposition 2 contains all the integers for which the above equations can have solutions in K^*.

Together with the algorithms of [3, 9] for the denominator up to a power of t, the bounding methods of the previous sections, and the known algorithms for solving parameterized linear differential equations over $C(x)$ [1, 2, 3, 9], the above algorithm proves the following result:

Theorem 2 *Let C be any field, and $E = C(x,t)$ where t is transcendental and exponential over $(C(x), d/dx)$. If we know a vector space basis for C over $\mathbb{Q}$ and if we can factor polynomials in $C[Z]$ and compute their integer roots, then we can effectively solve parameterized linear differential equations over E.*

Note that the requirement of factoring over an algebraic extension of C appears only in one branch of the bounding method, and only when the nonlinear system considered for γ and ν has strictly positive dimension. In particular, this cannot happen when $C = \mathbb{Q}$.

5 Examples

Consider first $\mathbb{Q}(x, e^{\int x+1/x}) = \mathbb{Q}(x,t)$ where $t'/t = x + 1/x$, which is not the logarithmic derivative of a $\mathbb{Q}(x)$-radical, and look for solutions $y \in \mathbb{Q}(x)[t, t^{-1}]$ of $Ly = 0$ where

$$\begin{aligned}L &= ((x^4 + 2x^2)t^2 - x^3 - x)\partial^2 \\ &\quad + (x^4 + 3x^2 - (3x^5 + 11x^3 + 6x)t^2)\partial \\ &\quad + (2x^6 + 9x^4 + 11x^2 + 6)t^2\,.\end{aligned}$$

- Bounding the degree and order of y:
  - Bounding the degree: $\deg_t(L) = 2$ so $L_2(at^{\deg_t(y)}) = 0$ for some $a \in \mathbb{Q}(x)$ where

$$\begin{aligned}L_2 &= (x^4 + 2x^2)\partial^2 - (3x^5 + 11x^3 + 6x)\partial \\ &\quad + 2x^6 + 9x^4 + 11x^2 + 6\,.\end{aligned}$$

We have $t'/t = x + 1/x$ whose polynomial part has degree 1, so $Q(\deg_t(y)) = 0$ where

$$Q = Z^2 - 3Z + 2 = (Z-1)(Z-2)$$

so $\deg_t(y) \le 2$.

– Bounding the order: $\nu_t(L) = 0$ so $L_0(at^{\nu_t(y)}) = 0$ for some $a \in \mathbb{Q}(x)$ where

$$L_0 = -(x+x^3)\partial^2 + (x^4+3x^2)\partial\,. \qquad (8)$$

As before, the polynomial part of t'/t is x, so

$$Q = -Z^2 + Z = Z(1-Z)$$

so $\nu_t(y) \ge 0$.

- Computing the solution: from the bounding step, we conclude that any solution $y \in \mathbb{Q}(x)[t,t^{-1}]$ of $Ly = 0$ must be of the form $y = y_0 + y_1 t + y_2 t^2$.

  – Reducing modulo t, we have $L_0 y_0 = 0$ where L_0 is given by (8). Recursively solving that equation over $\mathbb{Q}(x)$ we find that $y_0 = c_1 \cdot 1$ where c_1 is an arbitrary constant. We have

$$\frac{1}{t}(L - L_0)1 = (2x^6 + 9x^4 + 11x^2 + 6)t\,,$$

  so, replacing y by $c_1 + tz$ where $z = y_1 + y_2 t$ we get the new equation

$$\tilde{L}z = -c_1(2x^6 + 9x^4 + 11x^2 + 6)t$$

  where

$$\begin{aligned}\tilde{L} &= ((x^4+2x^2)t^2 - x^3 - x)\Big(\partial + x + \frac{1}{x}\Big)^2 \\ &+(x^4 + 3x^2 - (3x^5 + 11x^3 + 6x)t^2)\Big(\partial + x + \frac{1}{x}\Big) \\ &+(2x^6 + 9x^4 + 11x^2 + 6)t^2 \\ &= ((x^4+2x^2)t^2 - x^3 - x)\partial^2 \\ &\quad -((x^5 + 5x^3 + 2x)t^2 + x^4 + x^2 + 2)\partial\end{aligned}$$

  – Reducing modulo t, we have $\tilde{L}_0 y_1 = 0$, where

$$\tilde{L}_0 = -(x^3+x)\partial^2 - (x^4 + x^2 + 2)\partial\,.$$

  Recursively solving that equation over $\mathbb{Q}(x)$ we find that $y_1 = c_2 \cdot 1$ where c_2 is an arbitrary constant. We have $1/t(\tilde{L} - \tilde{L}_0)1 = 0$, so, replacing z by $c_2 + ty_2$ we get the new equation

$$\big(t^2\overline{L}_2 + \overline{L}_0\big)y_2 = -c_1(2x^6 + 9x^4 + 11x^2 + 6) \quad (9)$$

  where

$$\overline{L}_2 = (2x^2 + x^4)\partial^2 + (x^5 + x^3 + 2x)\partial - x^4 - x^2 - 2\,,$$

  and

$$\begin{aligned}\overline{L}_0 &= -(x^3+x)\partial^2 - (3x^4 + 5x^2 + 4)\partial \\ &\quad - \frac{2x^6 + 6x^4 + 6x^2 + 2}{x}\,.\end{aligned}$$

  – Equating the coefficients of t^2 and t^0 on both sides of (9) we get the equations $\overline{L}_2 y_2 = 0$ and

$$\overline{L}_0 y_2 = -c_1(2x^6 + 9x^4 + 11x^2 + 6)\,.$$

  Recursively solving those equations over $\mathbb{Q}(x)$ we find that $y_2 = c_3 x$ with the linear constraint $c_3 - c_1 = 0$. Therefore, the general solution of $Ly = 0$ is

$$y = c_1\Big(1 + xe^{2\int x+1/x}\Big) + c_2 e^{\int x+1/x}$$

  where c_1 and c_2 are arbitrary constants.

We illustrate in the second example the simpler version of the algorithm mentioned at the end of section 4: consider the differential equation

$$\frac{d^2y}{dx^2} - xy = (4x^3 - x^2 + 6x)e^{x^2+c} \qquad (10)$$

where c is an arbitrary parameter. Since the homonogeneous equation has no Liouvillian solutions [8], the inhomogeneous one has Liouvillian solutions if and only it has a solution $y \in \mathbb{Q}(c)(x, e^{x^2+c}) = C(x,t)$ where $C = \mathbb{Q}(c)$ and $t'/t = 2x$ [6]. Since the coefficients of the operator and the denominator of the right–hand side do not involve t, we only need to look for solutions $y \in C(x)[t,t^{-1}]$.

- Bounding the degree and order of y: we get $Q(\gamma) = 0$ where $Q = 4Z^2$, which implies that $\beta(L) = \{0\}$. Proposition 1 then implies that $\nu_t(y) \ge 0$ and $\deg_t(y) \le 1$.
- Computing the solution: from the bounding step, we conclude that any solution $y \in C(x)[t,t^{-1}]$ must be of the form $y = y_0 + y_1 t$. The equations (7) become $d^2y_0/dx^2 - xy_0 = 0$, which has no solution in $C(x)$, and

$$\left(\Big(\frac{d}{dx} + 2x\Big)^2 - x\right)y_1 = 4x^3 - x^2 + 6x\,.$$

  Recursively solving that equation over $C(x)$, we find the unique solution $y_1 = x$, so the only Liouvillian particular solution of (10) is $y = xe^{x^2+c}$.

6 Conclusion

We have presented a variant of Singer's algorithm for the "base case": simple exponential extensions of $(C(x), d/dx)$. Our method is effective and has been implemented[3]. Since the steps for finding the denominator and finding the solutions with known bounds work over arbitrary exponential extensions, the only step specific to the base case in our algorithm is the bounding method of Proposition 2. It turns out that if we replace $C(x)$ by a Liouvillian (or monomial) transcendental extension $K(\theta)$ of a differential field K (*i.e.* $E = K(\theta,t)$), then the algorithm of Proposition 2 can still be applied if we replace the hypothesis $t'/t - rf'/f \ne 0$ for $r \in \mathbb{Q}$ and $f \in C(x)^*$ by the stronger one $t'/t - rf'/f \notin K$ for $r \in \mathbb{Q}$ and $f \in K(\theta)^*$. Work on the details is under way and will be presented in a subsequent paper. We remark that together with the results of [5, 10], our algorithm yields a Liouvillian solver for completely reducible second order linear ordinary differential equations with coefficients in $C(x, e^{\int f(x)dx})$.

[3] `http://www.inria.fr/cafe/Manuel.Bronstein/bernina.html`

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